Soluzione di Equazioni e Sistemi di Equazioni Non Lineari: il metodo Newton-Raphson

Nonlinear Equations – Iterative Methods

Inizia da una soluzione di tentativo (guess) x^0

Genera una sequenza di iterazioni x^{n-1}, x^n, x^{n+1} che (si spera…) converga verso la soluzione cercata x^*

Le iterazioni sono generate in accordo con una funzione di iterazione F: $x^{n+1}=F(x^n)$

Domande:

- Dopo quante iterazioni la sequenza converge ?
- Quale è il convergence rate ?

Newton-Raphson (NR)

Fa parte degli "slope methods" e prevede la linearizzazione dell'equazione: la f(x) è linearizzata e risolta.

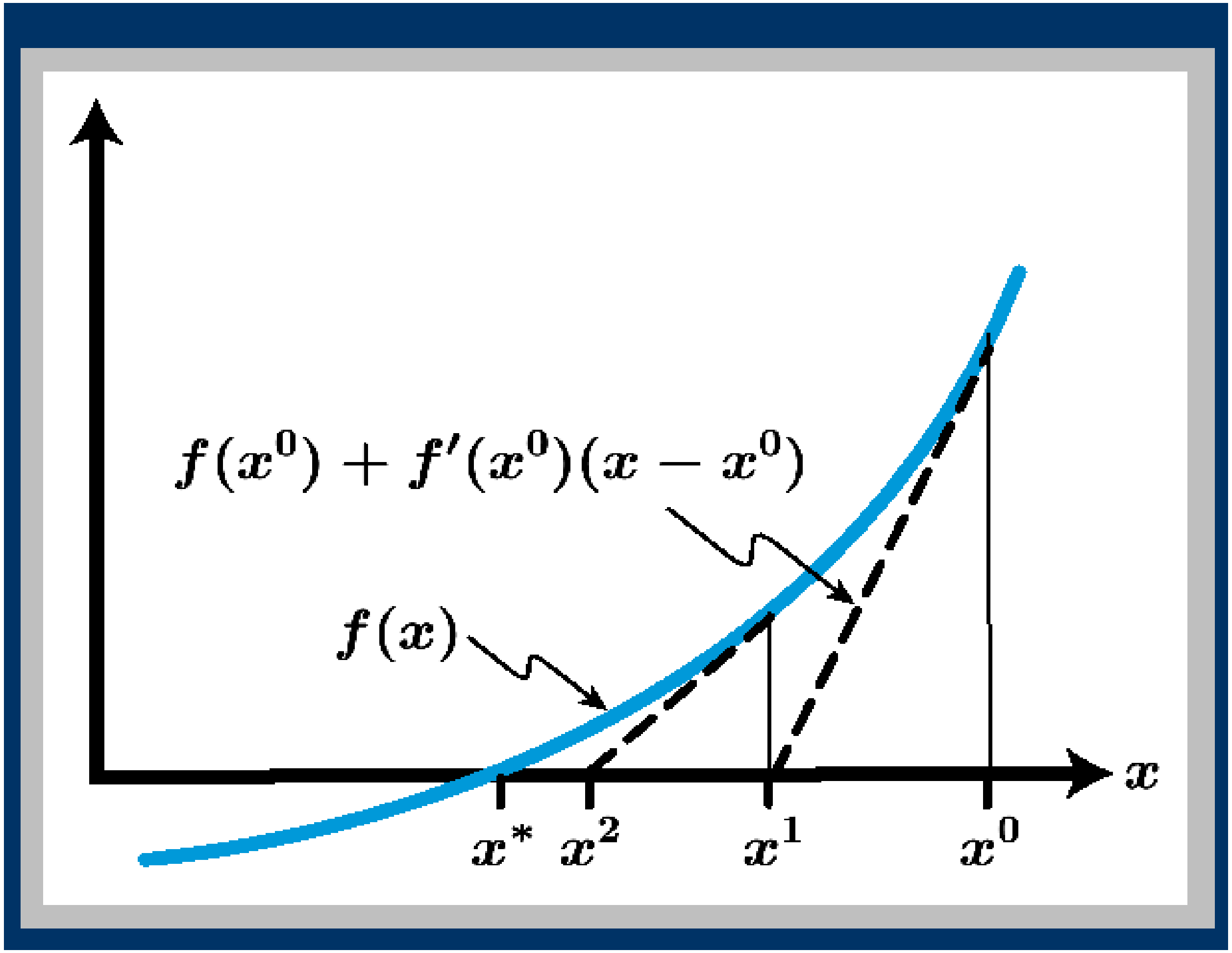
$$f(x) = f(x^*) + \frac{df}{dx}(x^*) \cdot (x - x^*) \qquad \text{Taylor Series}$$

$$f(x^{k+1}) = f(x^k) + \frac{df}{dx}(x^k) \cdot (x^{k+1} - x^k)$$

$$\Rightarrow x^{k+1} = x^k - \left[\frac{df}{dx}(x^k)\right]^{-1} \cdot f(x^k) \qquad \text{Iteration function}$$

Nota: ad ogni passo dell'iterazione occorre valutare sia *f* che *f'*

Newton-Raphson - Graficamente...



Newton-Raphson - Algoritmo

Do $k = 0$ to ....

$$x^{k+1} = x^k - \left[\frac{df}{dx}(x^k) \right]^{-1} f(x^k)$$

until *raggiunta convergenza*

Convergenza: l'iterazione $\{x^{(k)}\}$ converge con ordine q se esiste una norma tale che per ogni $k \geq N$:

$$\left\| x^{k+1} - \hat{x} \right\| \leq \left\| x^k - \hat{x} \right\|^q$$

Newton-Raphson - Convergenza

$$0 = f(x^*) = f(x^k) + \frac{df}{dx}(x^k)(x^* - x^k) + \frac{d^2 f}{dx^2}(\tilde{x})(x^* - x^k)^2$$

$$\text{some } \tilde{x} \in [x^k, x^*]$$

Il teorema del valor medio consente di troncare la serie di Taylor

Tuttavia, come da definizione NR :

$$0 = f(x^k) + \frac{df}{dx}(x^k)(x^{k+1} - x^k)$$

Newton-Raphson - Convergenza

Sottraendo $$\frac{df}{dx}(x^k)(x^{k+1}-x^*)=\frac{d^2f}{d^2x}(\tilde{x})(x^k-x^*)^2$$

Dividendo $$(x^{k+1}-x^*)=[\frac{df}{dx}(x^k)]^{-1}\frac{d^2f}{d^2x}(\tilde{x})(x^k-x^*)^2$$

Let $$\left|[\frac{df}{dx}(x^k)]^{-1}\frac{d^2f}{d^2x}(\tilde{x})\right|=K^k$$

Convergenza quadratica

then $$\left|x^{k+1}-x^*\right|\leq K^k\left|x^k-x^*\right|^2$$

Newton-Raphson - Convergenza

Convergenza Locale

Se il rapporto:

$$\left|\frac{f''(x)}{f'(x)}\right| < \infty$$

NR converge in modo quadratico, a patto che la soluzione iniziale sia vicina alla soluzione cercata (se nell'intervallo di ricerca la derivata ha un punto di nullo, la convergenza non è assicurata)

Newton-Raphson - Convergenza

La convergenza NON è quadratica se lo zero è a molteplicità m≥2

$$f(x) = \left(x - x^*\right)^m \cdot h(x)$$

$$f'(x) = \left(x - x^*\right)^{m-1} \cdot \left[m \cdot h(x) + \left(x - x^*\right) \cdot h'(x)\right]$$

$$x_{k+1} - x^* = x_k - x^* - \frac{\left(x_k - x^*\right)^m \cdot h(x_k)}{\left(x_k - x^*\right)^{m-1} \cdot \left[m \cdot h(x_k) + \left(x_k - x^*\right) \cdot h'(x_k)\right]}$$

$$\frac{x_{k+1} - x^*}{x_k - x^*} = 1 - \frac{h(x_k)}{m \cdot h(x_k) + \left(x_k - x^*\right) \cdot h'(x_k)}$$

$$\lim_{k \to \infty} \left[1 - \frac{h(x_k)}{m \cdot h(x_k) + \left(x_k - x^*\right) \cdot h'(x_k)}\right] = 1 - \frac{1}{m}$$

La convergenza è lineare

Newton-Raphson Method - Convergence

$x^0 =$ Initial Guess, $k = 0$

Repeat {

$$\frac{\partial f\left(x^k\right)}{\partial x}\left(x^{k+1} - x^k\right) = -f\left(x^k\right)$$

$$k = k + 1$$

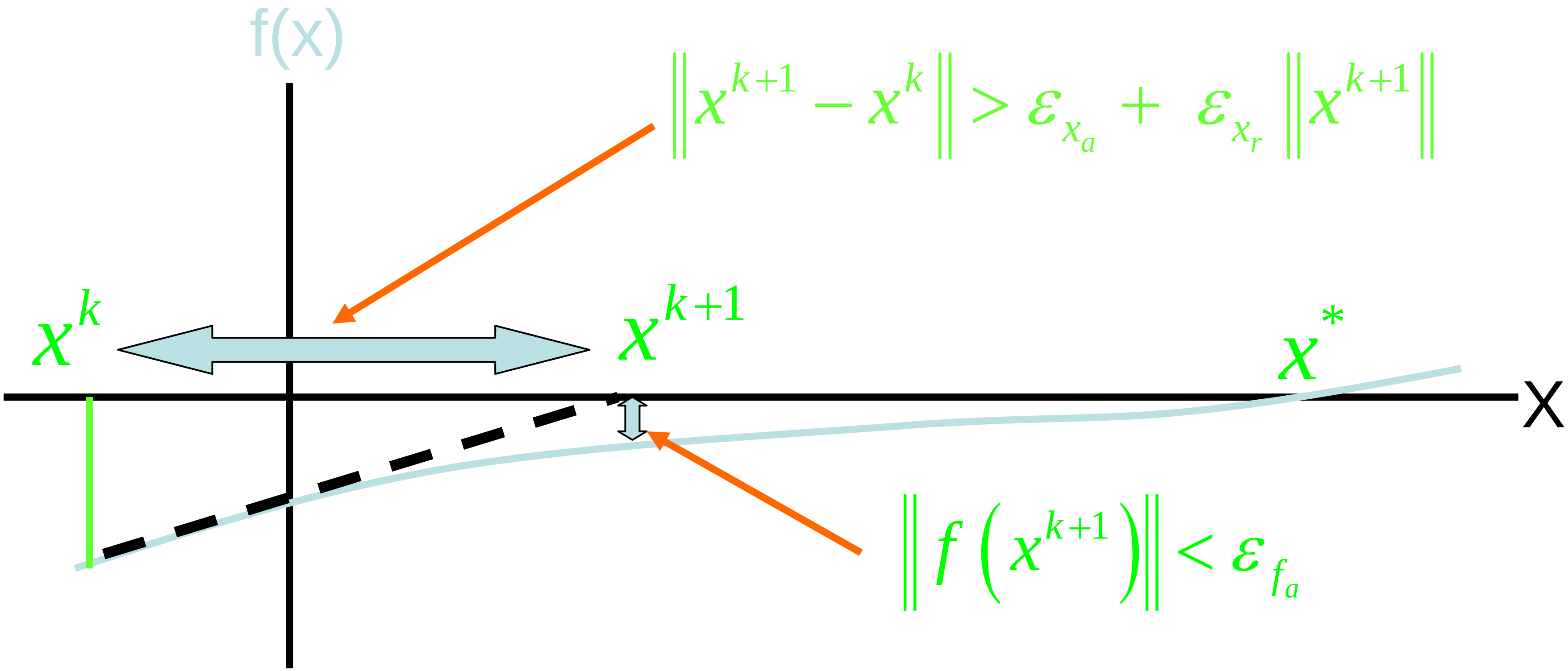
} Until ?

$\left\| x^{k+1} - x^k \right\| < threshold\,?$ $\left\| f\left(x^{k+1}\right) \right\| < threshold\,?$

Newton-Raphson Method - Convergence

Convergence Checks

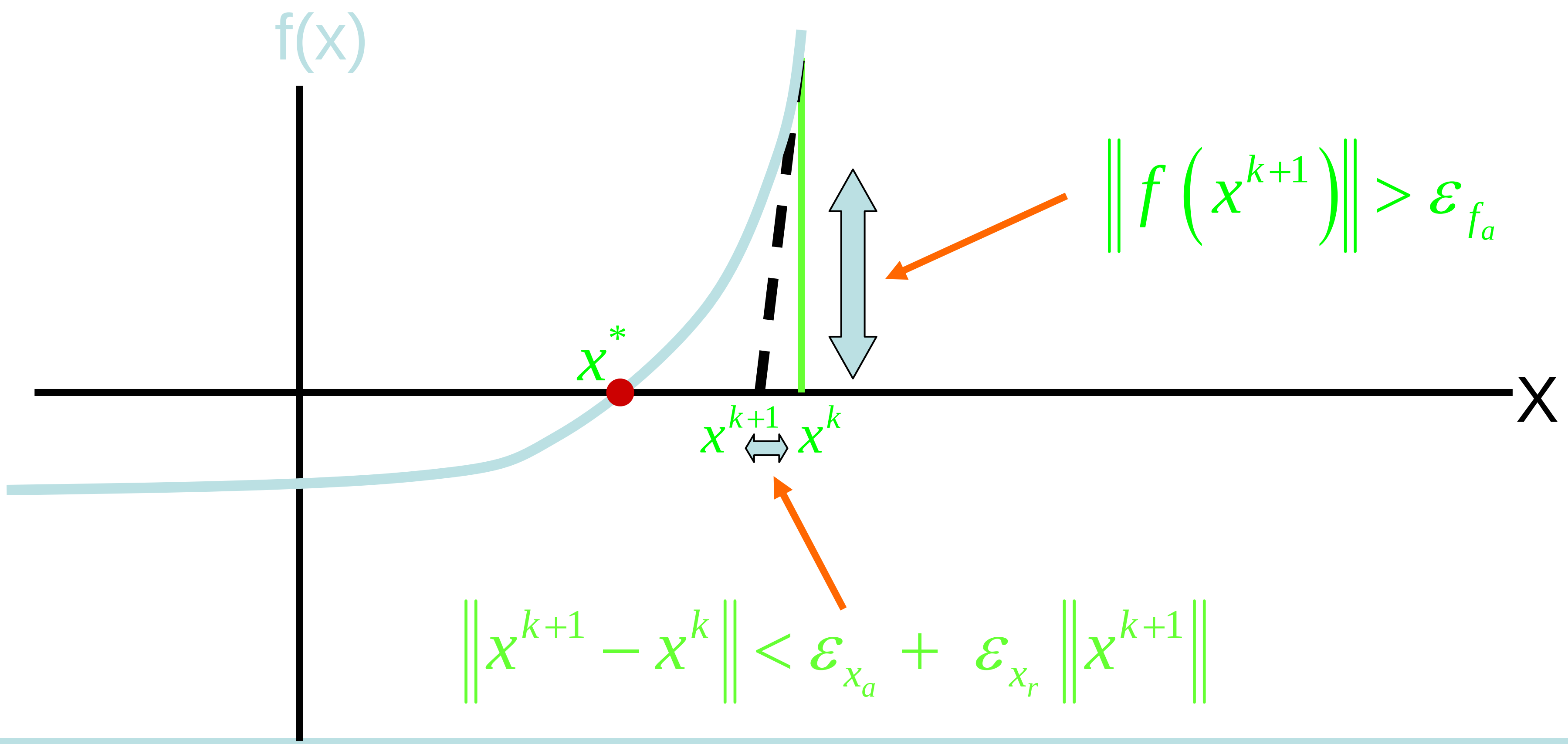
Need a "delta-x" check to avoid false convergence



Newton-Raphson Method - Convergence

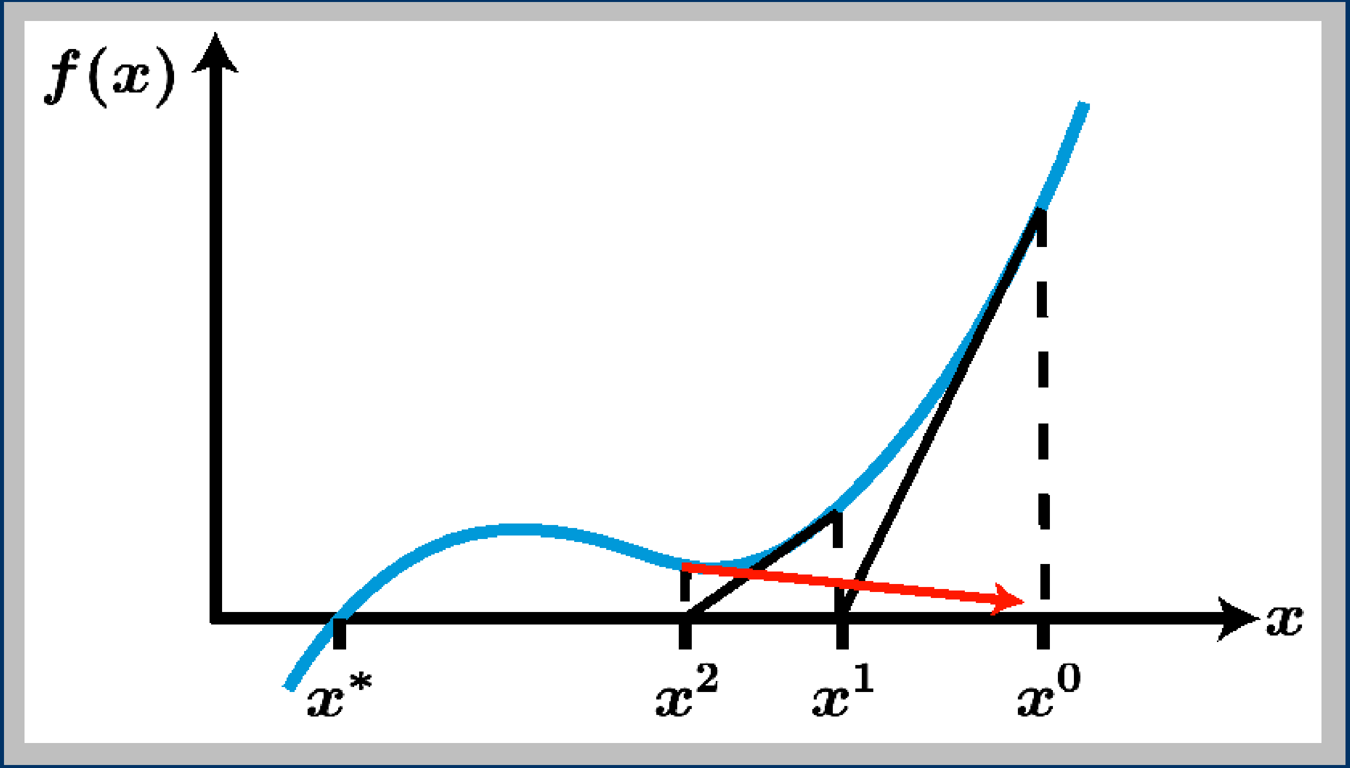
Convergence Checks

Also need an "$f(x)$" check to avoid false convergence



Newton-Raphson Method - Convergence

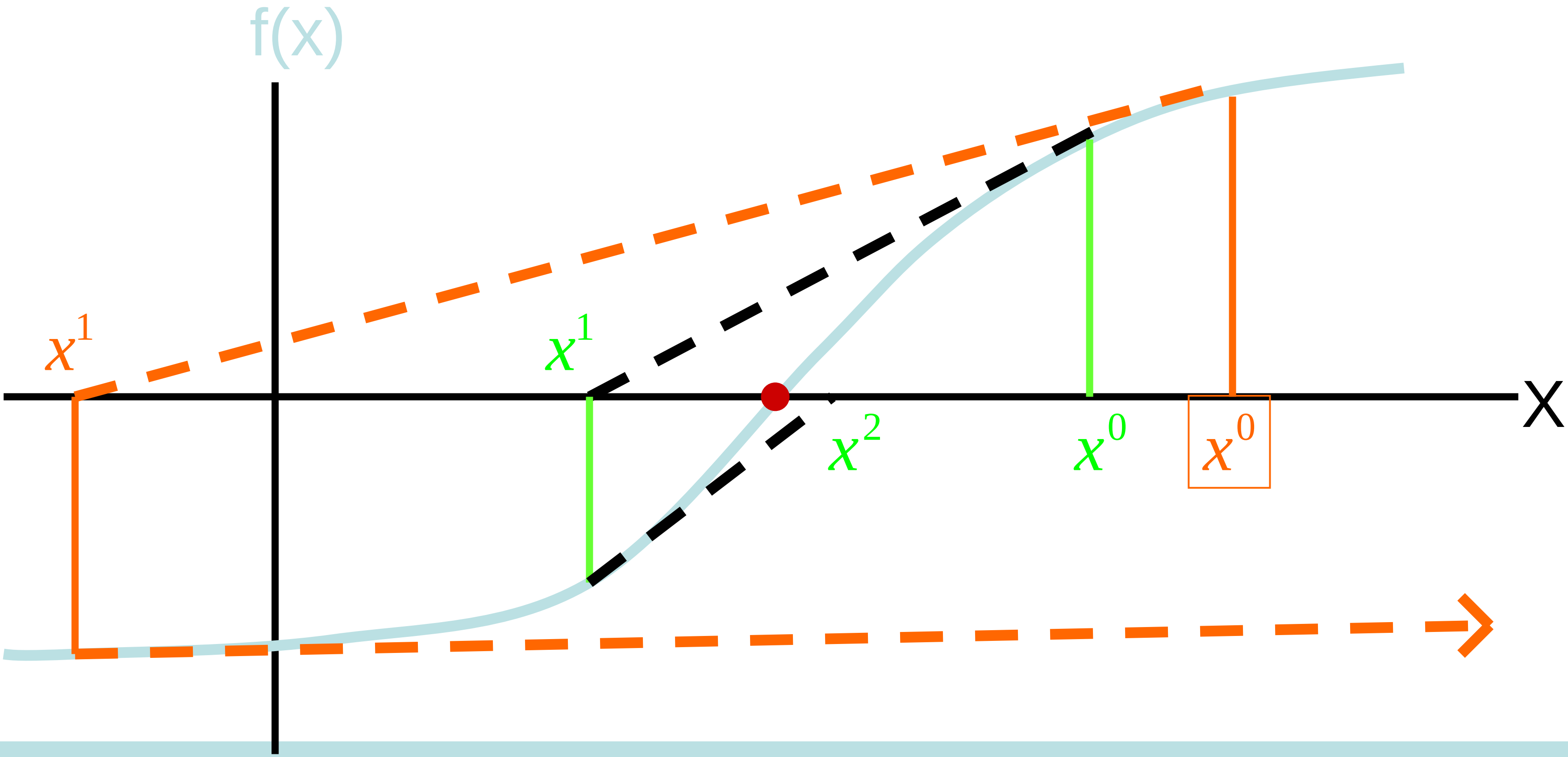
We require that x^0 be **"close"** to the solution x^*



Newton-Raphson Method - Convergence

Local Convergence

Convergence Depends on a Good Initial Guess

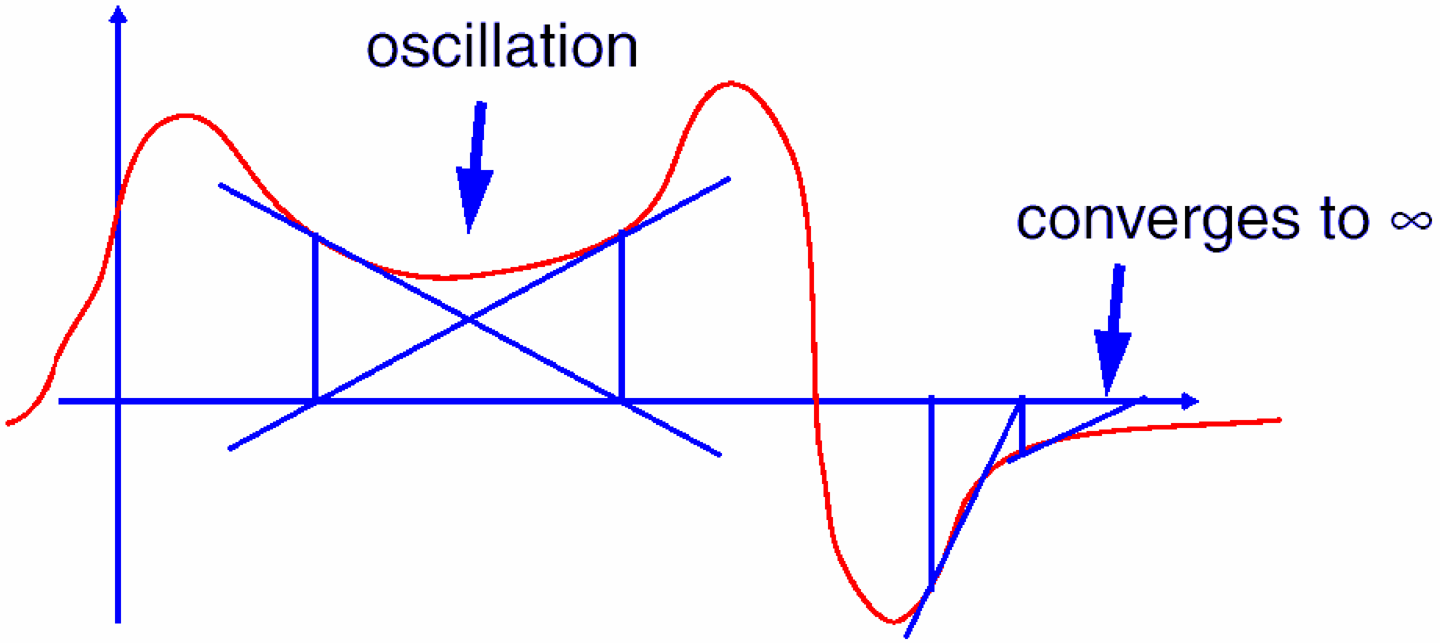


Newton-Raphson Method - Convergence

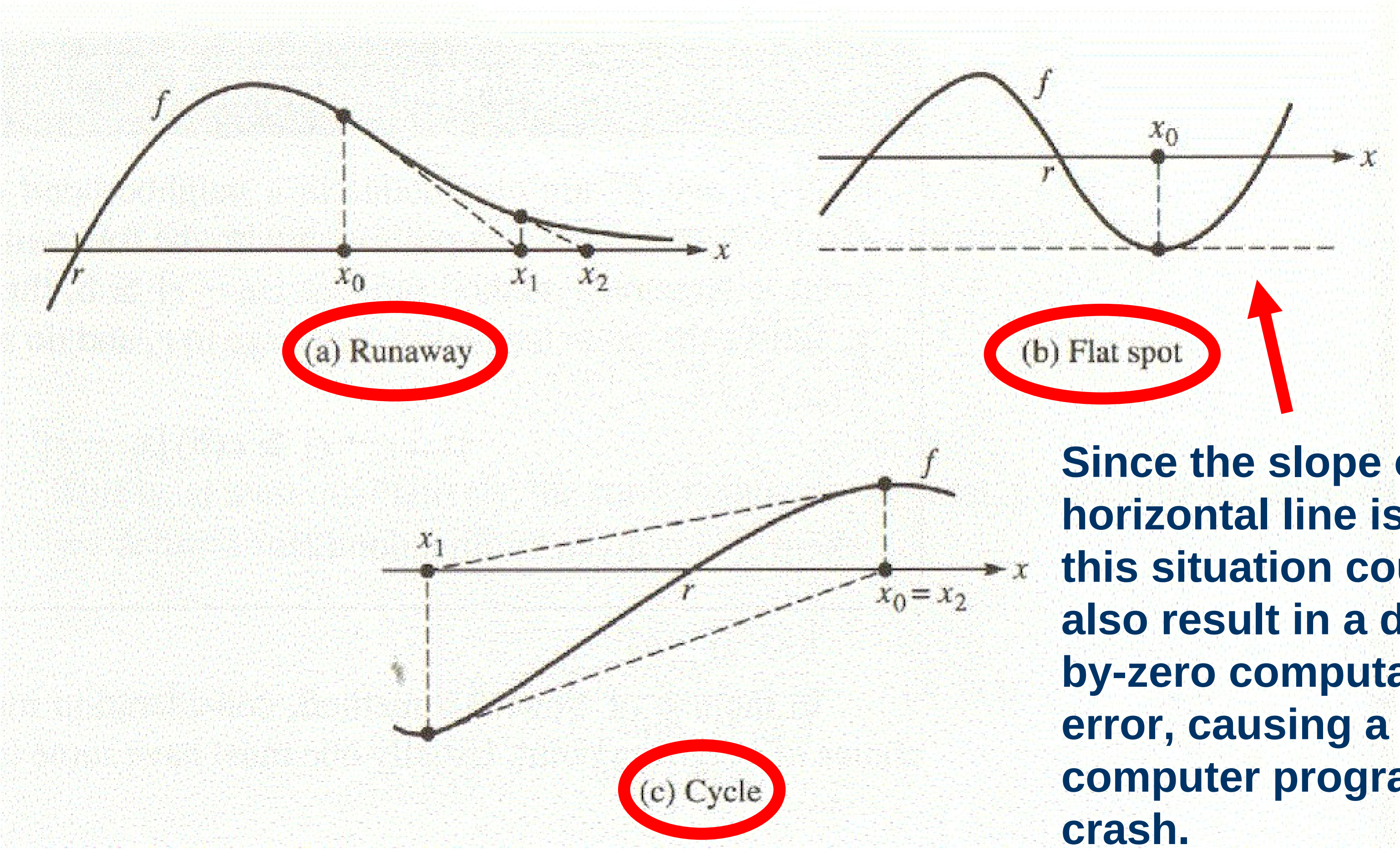
Local Convergence

Convergence Depends on a Good Initial Guess

Example:



Condizioni di criticità



Since the slope of a horizontal line is zero, this situation could also result in a divide-by-zero computational error, causing a computer program to crash.

Multidimensional Newton Method

Problem: Find x^* such that $F\left(x^*\right)=0$

$$\mathrm{x}^* \in \Re^{\mathrm{N}} \qquad \mathrm{F}:\Re^{\mathrm{N}} \rightarrow \Re^{\mathrm{N}}$$

$$F(x)=F(x^*)+J(x^*)(x-x^*)$$ *Taylor Series*

$$J(x)=\begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \cdots & \frac{\partial F_1(x)}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_N(x)}{\partial x_1} & \cdots & \frac{\partial F_N(x)}{\partial x_N} \end{bmatrix}$$ *Jacobian Matrix*

$$\Rightarrow x^{k+1}=x^k-J(x^k)^{-1}F(x^k)$$ *Iteration function*

Multidimensional Newton Method

Computational Aspects

$$Iteration: x^{k+1} = x^k - J(x^k)^{-1} F(x^k)$$

Do not compute $J(x^k)^{-1}$ (it is not sparse).

Instead solve: $J(x^k)(x^{k+1} - x^k) = -F(x^k)$

Each iteration requires:

Evaluation of $F(x^k)$

Computation of $J(x^k)$

Solution of a linear system of algebraic equations whose coefficient matrix is $J(x^k)$ and whose RHS is $-F(x^k)$

Multidimensional Newton Method

Algorithm

$x^0 =$ Initial Guess, $k = 0$

Repeat {

Compute $F\left(x^k\right), J_F\left(x^k\right)$

Solve $J_F\left(x^k\right)\left(x^{k+1} - x^k\right) = -F\left(x^k\right)$ for x^{k+1}

$k = k + 1$

} Until $\left\|x^{k+1} - x^k\right\|$, $\left\|f\left(x^{k+1}\right)\right\|$ small enough

Multidimensional Newton Method Convergence

Local Convergence Theorem

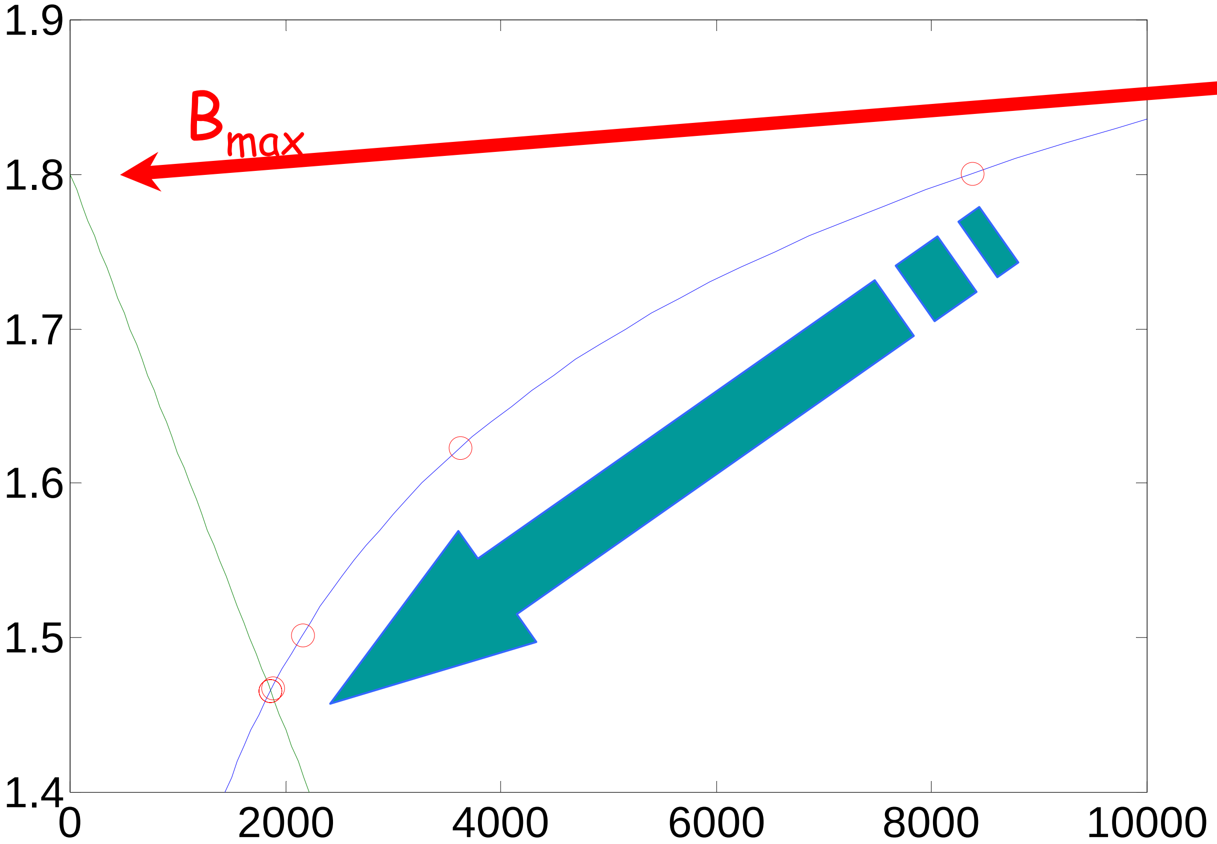
If

$a)\ \left\| J_F^{-1}\left(x^k\right)\right\| \le \beta \qquad (\text{Inverse is bounded})$

$b)\ \left\| J_F\left(x\right) - J_F\left(y\right)\right\| \le \ell \left\| x - y \right\| \qquad (\text{Derivative is Lipschitz Cont})$

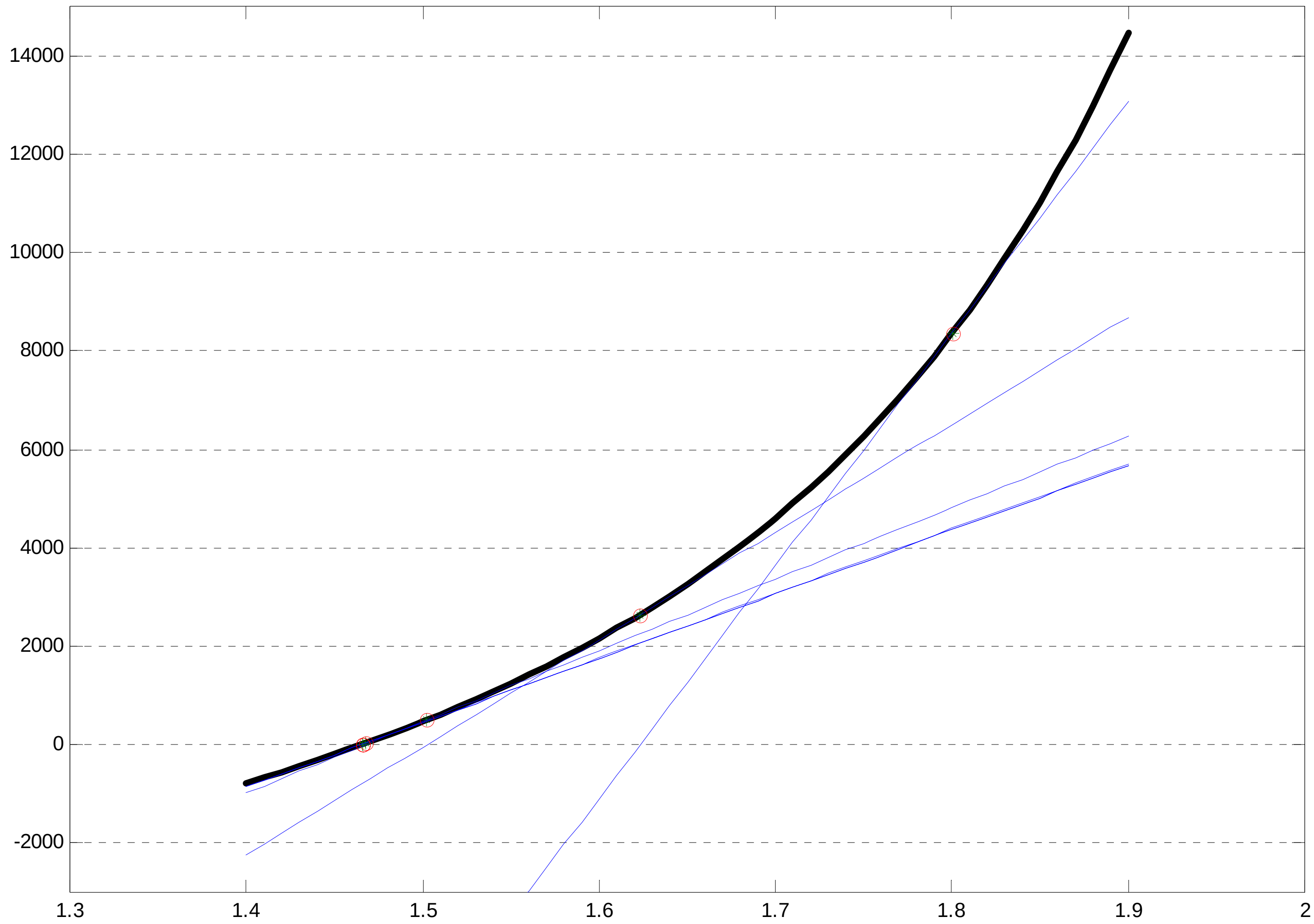
Then Newton’s method converges given a sufficiently close initial guess (and convergence is quadratic)

Applicazione all'elettromagnete: il processo iterativo



B(k)	Δ
1.8003	1.0000e2
1.6231	9.8449e-2
1.5022	7.4499e-2
1.4676	2.3039e-2
1.4656	1.3748e-3
1.4655	1.0181e-5

Le tangenti



14000
12000
10000
8000
6000
4000
2000
0
-2000
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2